On the construction of global models describing rotating bodies; uniqueness of the exterior gravitational field

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Abstract

The problem of constructing global models describing isolated axially symmetric rotating bodies in equilibrium is analyzed. It is claimed that, whenever the global spacetime is constructed by giving boundary data on the limiting surface of the body and integrating Einstein's equations both inside and ouside the body, the problem becomes overdetermined. Similarly, when the spacetime describing the *interior* of the body is explicitly given, the problem of finding the *exterior* vacuum solution becomes overdetermined. We discuss in detail the procedure to be followed in order to construct the exterior vacuum field created by a given but arbitrary distribution of matter. Finally, the uniqueness of the exterior vacuum gravitational field is proven by exploiting the harmonic map formulation of the vacuum equations and the boundary conditions prescribed from the matching.

The description and analysis of isolated rotating objects in equilibrium is a fundamental problem in general relativity. Unfortunately, despite important efforts in that direction, our present understanding of the question is still very incomplete. In particular, not a single model describing both the interior and the exterior of a self-gravitating, spatially compact, rotating object is explicitly known. There are, however, very interesting solutions which describe the exterior field of a physical object. We are referring to the solution for the rotating disc of dust recently found by Neugebauer

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and Mainel [1]. This metric is vacuum everywhere and contains discontinuities in the first derivatives of the metric across an equatorial disc centered on the axis of symmetry, which can be interpreted as a shell of dust. Despite its manifest interest and physical relevance, this solution does not contain an interior body in strict sense (in mathematical terms, the interior of the body is empty). In general, solving the vacuum field equations everywhere and allowing for discontinuities in the first derivatives of the metric will be adequate for describing the gravitational field generated by a shell of matter, or as a possibly good approximation of some very thin objects, but it cannot be used as an exact solution when the source object has a non-empty interior. In that case, the problem becomes even more difficult because the global model must be constructed from two spacetimes, one describing the interior of the body (i.e. a solution of Einstein's field equations with matter) and another one describing the exterior field (i.e. an asymptotically flat solution of the vacuum equations). Several methods have been presented recently for attacking this kind of problem. One of them requires the resolution of Einstein's field equations both inside and outside the body starting from a common boundary. This approach is taken, for instance, in a recent paper [2] where the existence, regularity and uniqueness of the solution has been investigated. More precisely, the authors prescribe Dirichlet boundary data on a two-sphere and then show that both the vacuum field equations and the interior problem (assuming rigid rotation) are solvable. The interest of these results is obvious, but it must be noted that they still *do not* prove the solvability of the global problem, not even for the rigidly rotating case. The reason is that it cannot be ensured that the two solutions thus constructed match appropriately across the boundary hypersurface. Given the type of boundary data used in [2], it follows that the global spacetime has indeed a continuous metric, but nothing can be said a priori about the first derivatives of the metric (they can certainly be discontinuous). Hence, in general, the global spacetime will have an undesired and uncontrolled surface layer of matter, which is an important weakness of the method. The same argument shows in general that, whenever two different spacetime pieces (interior and exterior) are used to construct the global manifold, the physical problem requires imposing the *full* set of matching conditions and not just a (mathematically suitable) subset of them.

In order to solve the global problem one can alternatively adopt the point of view of finding the whole spacetime once and for all. The most favourable situation would be finding the equivalent of a Green function for the coupled system of equations describing gravity and matter, so that the solution can be expressed directly in terms of the sources. Although this is probably the mathematically most well-motivated method, the technical difficulties are enormous and very scarce results are known at present. In particular, it seems very unlikely that an explicit model can be obtained using this method in the short, or even in the medium, term. It must be mentioned, however, that this approach has been followed in a very interesting paper by Heilig [3], where the existence of global solutions describing rigidly rotating stars is proven for configurations sufficiently close to a Newtonian solution. A second global method that can be employed requires finding smooth solutions of the differential system describing gravity

and matter, such that the matter fields tend to zero fast enough when we approach infinity (so that the solution can still be interpreted as an isolated body). Although this method is potentially capable of producing explicit solutions, its main drawback is that the rotating body is not truly finite in size because it has no limiting surface. Hence, the solutions have no vacuum exterior. Of course this does not mean that the solutions are necessarily unphysical, but for many purposes it is highly convenient to have models in which the rotating object has a clear limiting boundary. A second disadvantage shared by the two global methods described above is that the system of differential equations to be analysed must be closed, and this requires the imposition of conditions on the type of matter beforehand such as, for instance, rigid rotation and a barotropic equation of state. This means that a different analysis must be performed for each different type of matter.

All in all, it follows that for any truly isolated and finite body, the full set of matching conditions between the interior metric and the exterior vacuum solution must be inevitably imposed in order to analyse the physical problem. Excluding the "Green function" approach, the construction of the global solution requires three steps, namely, obtaining the interior metric (either by solving the Einstein field equations for an adequate type of matter or by any other means), solving the exterior vacuum field equations and imposing the matching conditions between the two spacetimes. These three steps could be solved, in principle, in any order. In our opinion, however, the natural ordering is to provide first the interior metric and then finding the exterior metric which satisfies the matching conditions with the prescribed interior region. This point of view incorporates in a natural way the common idea that the exterior field is created by the interior distribution of matter. Moreover, the natural questions of existence and uniqueness of the gravitational field are only reasonable for the exterior problem, once the interior metric is known. The reason is that we cannot expect existence, or even uniqueness, of the interior region for a given exterior gravitational field (it is well-known that in Newtonian gravity different distributions of matter produce the same exterior gravitational field; also, the spherically symmetric case proves that this still holds in General Relativity). Moreover, the inside-outside procedure has the important advantage that the resolution of the interior field equations, which are much more difficult to handle that the vacuum equations, can be performed by any method we wish, or even not done at all if we prescribe the interior metric by hand. In other words, there is no need to solve the interior equations in order to study the associated exterior problem. Obviously, for physically relevant problems the interior field equations should be solved, but the important point here is that the interior problem is completely decoupled from the exterior one.

The aim of this letter is, besides stressing and clarifying all the points above, to describe in detail the conditions one obtains by matching an interior (arbitrary but supposed to be known) stationary and axially symmetric metric with an unknown exterior vacuum metric, as well as to analyze and solve some of the problems and difficulties that arise. In particular, we show that *the junction conditions introduce two new essential parameters in the interior* metric. To the best of our knowledge, this

fact was unknown hitherto. We also show that, for each value of these two parameters, the matching hypersurface, both from the exterior and the interior regions, is uniquely determined. Furthermore, the matching conditions fix the boundary conditions for the exterior problem. When written in terms of the Ernst potential [4], the boundary value exterior problem turns out to be *overdetermined* and *not unique*. It is overdetermined because the interior geometry of the self-gravitating object provides more boundary conditions than necessary for solving the Ernst equations and it is not unique for, as we shall see, the boundary conditions fix the Ernst potential on the matching hypersurface up to an arbitrary, non-trivial, additive constant. Since the Einstein field equations for stationary fields are of elliptic type, it follows that the physical problem is overdetermined and mathematically not well-posed. After computing and classifying the matching conditions, we finally prove the *uniqueness* of the exterior vacuum gravitational field generated by a given isolated, axially symmetric, rotating object in equilibrium. Although this result is quite natural and it has been often assumed implicitly in numerical calculations, no proof is available in the literature. The only results we are aware of are due to Lichnerowicz [5], but they only apply doubly locally, in the sense that they hold only in a neighbourhood of a local piece of the matching hypersurface. The method we use for proving the uniqueness rely on the harmonic map formulation of the exterior vacuum field equations. This provides a proof which is extremely simple and which is valid for *any* boundary data (by using more classical methods of functional analysis one normally obtains theorems which only hold for boundary data satisfying certain inequalities). It must be emphasized here that none of the recent results on the uniqueness of solutions of the Ernst equations – see e.g. [6] and references therein –, analyzes the problem we are considering in this letter.

Let us then describe in detail the procedure discussed above for constructing global models for rotating objects. Let us assume we are given a stationary and axisymmetric spacetime (V_I, g_I) describing the interior of the selfgravitating fluid. We make the usual assumption that this metric satisfies the circularity condition [7] (for fluids without energy flux, this is equivalent [8] to the absence of convective motions). Hence, there exist coordinates $\{T, \Phi, r, \zeta\}$ in which the line-element reads

$$ds_I^2 = -e^{2V}\left(dT + Bd\Phi\right)^2 + e^{-2V}\left[e^{2h}\left(dr^2 + d\zeta^2\right) + \alpha^2 d\Phi^2\right] \qquad (1)$$

where V, B, h and α are (known) functions of r and ζ and ∂_Φ is the axial Killing. The searched vacuum exterior manifold (V_E, g_E) is assumed truly stationary (free of ergoregions and/or Killing horizons) so that it can be globally described using Weyl coordinates

$$ds_E^2 = -e^{2U}\left(dt + Ad\phi\right)^2 + e^{-2U}\left[e^{2k}\left(d\rho^2 + dz^2\right) + \rho^2 d\phi^2\right] \qquad (2)$$

where U, A and k are functions of ρ and z only. The axial Killing vector of this spacetime is ∂_ϕ and the axis of symmetry is located at $\rho = 0$. The coordinates have been chosen so that ∂_t is the unique timelike Killing vector which is unit at infinity (i.e. t mesures proper time at infinity) Then, the coordinate freedom in (2) consists only of

trivial constant shifts of t, ϕ and z. Introducing the Ernst potential [4] $\mathcal{E} \equiv e^{2U} + i\,\Omega$, where Ω is defined up to an additive constant by

$$\rho\,\Omega_{,\rho} = -e^{4U} A_{,z}\,, \qquad \rho\,\Omega_{,z} = e^{4U} A_{,\rho}\,, \tag{3}$$

(a comma indicates partial derivative), the vacuum field equations read simply

$$2\,\triangle U + e^{-4U}\langle d\Omega, d\Omega\rangle = 0\,, \qquad \triangle\Omega - 4\langle d\Omega, dU\rangle = 0\,, \tag{4}$$

where d is the exterior derivative, $\triangle$ is the laplacian of the 3-Euclidean metric

$$ds^2_M = d\rho^2 + dz^2 + \rho^2 d\phi^2$$

and $\langle\,,\,\rangle$ denotes scalar product in this flat space. The equation for the remaining function k is a simple quadrature integrable once the solution of (4) is obtained. Thus, the system of differential equations to be solved constitutes a coupled, non-linear, quasi-linear elliptic system of p.d.e. where the principal part is the Laplace-Beltrami operator on the Euclidean 3-space.

In order to find the boundary conditions of this problem, the matching of the known interior (V_I, g_I) with the undetermined exterior (V_E, g_E) must be analyzed. The standard junction theory involves finding two isometric imbeddings of a 3-dimensional manifold (Σ, g_Σ) into (V_I, g_I) and (V_E, g_E) respectively such that the two second fundamental forms in Σ inherited from both imbeddings coincide (see e.g. [9]). It is natural to assume that Σ preserves both the stationarity and the axisymmetry and hence local coordinates $\{\lambda, \tau, \varphi\}$ in Σ can be chosen so that the imbedding $\chi_E\,:\,\Sigma \longrightarrow V_E$ reads

$$\chi_E : \{t = \tau,\ \phi = \varphi,\ \rho = \rho(\lambda),\ z = z(\lambda)\}\,, \tag{5}$$

that is, we adapt the coordinates in Σ so that ∂_τ and ∂_φ coincide with ∂_t and ∂_ϕ on the exterior matching hypersurface $\chi_E(\Sigma)$. The next task is the determination of the imbedding $\chi_I\,:\,\Sigma \longrightarrow V_I$. Obviously, the images of ∂_τ and ∂_φ in $\chi_I\,(\Sigma)$ must be Killing vectors. Since the axial Killing vector has an intrinsic definition as the only one with 2π-periodic closed orbits and a symmetry axis, we must impose $\chi_{I*}\left(\partial_\varphi|_\Sigma\right) = \partial_\Phi|_{\chi_I(\Sigma)}$ (χ_{I*} is the push-forward of χ_I). Regarding the image of ∂_τ, and given that the stationary Killing vector in (V_I, g_I) does not have an intrinsic characterization, we must allow for

$$\chi_{I*}\left(\partial_\tau|_\Sigma\right) = a\,\left(\partial_T + b\partial_\Phi\right)|_{\chi_I(\Sigma)}\,, \tag{6}$$

where a is a positive constant (to preserve the time orientation) and b is a constant restricted only to preserve the timelike character of the Killing. In order to deal with this freedom, we perform in (V_I, g_I) the coordinate change

$$\Phi = \Phi' + abT', \qquad T = aT' \tag{7}$$

which implies $\partial_{T'} = a\left(\partial_T + b\partial_\Phi\right)$ and $\partial_{\Phi'} = \partial_\Phi$, leaving the axial Killing invariant. The new metric potentials (defined so that (1) keeps the same form when all unprimed quantities are substituted by their primed counterparts) are $\alpha' = a\alpha$, $h' - V' = h - V$,

$$e^{2V'} = a^2\left[\left(1 + bB\right)^2 e^{2V} - \alpha^2 b^2 e^{-2V}\right], \qquad B' = \frac{B\left(1 + bB\right)e^{2V} - \alpha^2 b e^{-2V}}{a\left(1 + bB\right)^2 e^{2V} - a\alpha^2 b^2 e^{-2V}}\,.$$

Thus, the matching procedure has introduced two new *essential* parameters in the interior metric (even though they are irrelevant for the interior geometry by itself). In order to clarify this, let us show that matchings with different values of a and b are essentially inequivalent from the exterior point of view, by taking as an example an interior describing a non-convective fluid with velocity vector $\vec{u} = N\left(\partial_T + w\partial_\Phi\right)$, where N and ω are functions of r and ζ. For any matching satisfying (6), $\vec{u}$ on the matching hypersurface as seen from the exterior becomes

$$\vec{u}|_\Sigma = N\left[\frac{1}{a}\partial_t + (w-b)\,\partial_\phi\right]\Big|_\Sigma \tag{8}$$

(we will identify $\Sigma \equiv \chi_I(\Sigma) \equiv \chi_E(\Sigma)$ from now on to simplify notation). The coordinates t and ϕ in the asymptotically flat exterior have an intrinsic meaning and (8) shows that the proper angular velocity of the fluid on Σ is $a\,(w-b)$, which depends essentially on a and b. Thus, different values of a and b will potentially give rise to different exteriors (whenever they exist), because the state of motion of the interior body is distinct as seen from infinity. Consequently, we should not expect that the exterior gravitational field for different values of a and b coincides, and the physically relevant problem is the uniqueness of the exterior gravitational field when a and b are fixed. In other words, the exterior gravitational field will be fixed when the interior metric *and* the identification of the interior with the exterior through Σ are both prescribed.

The new coordinate system (7) will be used from now on dropping primes everywhere. The imbedding $\chi_I : \Sigma \longrightarrow V_I$ reads then

$$\chi_I : \{T=\tau, \Phi=\varphi, r=r(\lambda), \zeta=\zeta(\lambda)\}\,. \tag{9}$$

The four unknowns $\rho(\lambda)$, $z(\lambda)$ in (5) and $r(\lambda)$, $\zeta(\lambda)$ in (9) define the junction hypersurfaces and must be determined from the matching conditions. It can be shown [10] that all the matching equations can be reorganized into the following three sets of conditions (to prove the equivalence the vacuum field equations for the exterior have to be used):
1 *Conditions on the interior hypersurface.* The functions $r(\lambda)$, $\zeta(\lambda)$ defining the interior hypersurface are the solutions of the overdetermined system of ordinary differential equations

$$G_{22}\dot{\zeta} - G_{23}\dot{r}\Big|_\Sigma = 0, \qquad G_{32}\dot{\zeta} - G_{33}\dot{r}\Big|_\Sigma = 0, \tag{10}$$

where the G's are the components of the Einstein tensor of (1) in the orthonormal tetrad

$$\boldsymbol{\theta}^{\mathbf{0}} = e^V\left(dT + Bd\Phi\right), \qquad \boldsymbol{\theta}^{\mathbf{1}} = \alpha e^{-V}d\Phi, \qquad \boldsymbol{\theta}^{\mathbf{2}} = e^{h-V}dr, \qquad \boldsymbol{\theta}^{\mathbf{3}} = e^{h-V}d\zeta,$$

and the dot denotes derivative with respect to λ. In many cases, these two differential equations are incompatible and no matching hypersurface exists, meaning that the given metric cannot be considered as a model for describing an isolated body with a vacuum exterior. There are also particular energy-momentum tensors for which the

above equations do not provide any information at all (those with $G_{22} = G_{23} = G_{33} = 0$, including the important case of dust), so that the matching can be performed, in principle, at any timelike hypersurface preserving the symmetry. However, as is clear from (10), the interior matching hypersurface is unique in the generic case.
2 *Exterior matching hypersurface.* The exterior matching hypersurface $\rho(\lambda), z(\lambda)$ is then uniquely determined by

$$\rho(\lambda) = \alpha|_{\Sigma} \,, \qquad \dot{z}\,(\lambda) = \alpha_{,r}\dot{\zeta} - \alpha_{,\zeta}\dot{r}\Big|_{\Sigma} \tag{11}$$

(the additive constant in $z\,(\lambda)$ is spurious given the shift freedom $z \longrightarrow z + \text{const.}$).
3 *Boundary conditions for the exterior problem.* Finally, the matching provides boundary conditions on the exterior metric potentials U and A at the matching hypersurface

$$\begin{aligned} U|_{\Sigma} &= V|_{\Sigma}\,, & A|_{\Sigma} &= B|_{\Sigma}\,, \\ U_{,\rho}\dot{z} - U_{,z}\dot{\rho}|_{\Sigma} &= V_{,r}\dot{\zeta} - V_{,\zeta}\dot{r}\Big|_{\Sigma}\,, & A_{,\rho}\dot{z} - A_{,z}\dot{\rho}|_{\Sigma} &= B_{,r}\dot{\zeta} - B_{,\zeta}\dot{r}\Big|_{\Sigma}\,. \end{aligned} \tag{12}$$

¿From these equations we must get the appropriate boundary conditions for the exterior vacuum problem in terms of the Ernst potential. This can be done by noticing that

$$\begin{aligned} \dot{\Omega}\Big|_{\Sigma} = \; \Omega_{,\rho}\dot{\rho} + \Omega_{,z}\dot{z}\Big|_{\Sigma} &= \frac{e^{4U}}{\rho}\left(-A_{,z}\dot{\rho} + A_{,\rho}\dot{z}\right)\Big|_{\Sigma} = \frac{e^{4V}}{\alpha}\left(B_{,r}\dot{\zeta} - B_{,\zeta}\dot{r}\right)\Big|_{\Sigma}\,, \\ \Omega_{,\rho}\dot{z} - \Omega_{,z}\dot{\rho}\Big|_{\Sigma} &= -\frac{e^{4U}}{\rho}\left(A_{,z}\dot{z} + A_{,\rho}\dot{\rho}\right)\Big|_{\Sigma} = -\frac{e^{4V}}{\alpha}\left(B_{,r}\dot{r} + B_{,\zeta}\dot{\zeta}\right)\Big|_{\Sigma}\,, \end{aligned}$$

where the righthand sides are known. Thus, the matching conditions fix the normal derivative of Ω on Σ uniquely, but they only fix Ω on Σ up to an arbitrary additive constant. As we shall see, this fact will add some subtleties in the uniqueness proof below.

We are left with Ernst equations (4) subject to boundary conditions both at the fixed (and known) compact axially symmetric surface given by (11) and at infinity. The conditions at infinity ensure that the spacetime is asymptotically flat (which is the mathematical translation of the isolation of the body) and are given by

$$U = -mR^{-1} + O(R^{-2}), \qquad \Omega = -2zJR^{-3} + O(R^{-3}) \tag{13}$$

where $R \equiv \sqrt{\rho^2 + z^2}$ and m and J are the total mass and angular momentum of the source, respectively. It is now evident that the boundary conditions on Σ constitute a Cauchy problem for the *elliptic* system (4), so that it is mathematically not well-posed. Thus, the physical problem is overdetermined and we should not expect existence of solutions in all situations. This is the main problem for proving the existence of the exterior solution given the interior, and it is a matter of further investigation to find which restrictions on (V_I, g_I) will allow to overcome it.

Nevertheless, we can now address the proof of the uniqueness of the exterior field, which is one of the purposes of this letter, by using the theory of harmonic maps.

Roughly speaking, harmonic maps are applications between Riemannian manifolds that extremize the so-called *energy or Dirichlet functional.* More precisely, given an n-dimensional Riemannian manifold (possibly with boundary) $(M, \overline{g})$, a k-dimensional Riemannian manifold (N, γ), and a C^1 map $\Psi : (M, \overline{g}) \longrightarrow (N, \gamma)$, we can construct the energy functional $E(\Psi) \equiv \int_M \frac{1}{2} \left[tr_M \Psi^\star(\gamma) \right] \boldsymbol{\eta}$, where $\boldsymbol{\eta}$ is the volume form of M and $tr_M \Psi^\star(\gamma)$ is the trace in M of the pull-back of the metric tensor γ of the target manifold N. If we introduce local coordinates $\{x^\mu\}$ and $\{\Psi^a\}$ in M and N respectively, then $E(\Psi)$ becomes

$$E(\Psi) = \int_M \frac{1}{2} \overline{g}^{\mu\nu}(x) \gamma_{ab}(\Psi(x)) \frac{\partial \Psi^a}{\partial x^\mu} \frac{\partial \Psi^b}{\partial x^\nu} \sqrt{\det \overline{g}}\, dx^1 \wedge \cdots \wedge dx^n$$

where μ, ν run from 1 to n and a, b run from 1 to k. Then, any C^2 map with finite $E\left(\Psi\right)$ is called harmonic iff it is a critical point of E with respect to variations vanishing on the boundary of M (whenever $\partial M \neq \emptyset$). Thus, any harmonic map Ψ satisfies the corresponding Euler-Lagrange equations, which read (in local coordinates for N)

$$\triangle \Psi^c + \Gamma^c_{ab}(\Psi(x)) \langle d\Psi^a, d\Psi^b \rangle = 0, \tag{14}$$

where Γ^c_{ab} are the Christoffel symbols in N. It is well-known that Ernst equations (4) can be written in this Euler-Lagrange form if the domain manifold M is chosen to be the part of the Euclidean 3-space with extends from the two-dimensional compact boundary $\tilde{\Sigma} \equiv \{\rho(\lambda), z(\lambda), \phi\}$ to infinity and the target manifold N to be the two-dimensional manifold with metric

$$ds^2_N = 4dU^2 + e^{-4U} d\Omega^2. \tag{15}$$

(Notice that $\tilde{\Sigma}$ is in fact the image of a new imbedding of the abstract Σ, in this case into M. Thus, even though there is a one-to-one correspondence between Σ and $\tilde{\Sigma}$, they should not be confused as their corresponding first fundamental forms do not coincide). The solutions of (4) we are interested in must must have the asymptotic behaviour (13) which ensures that the corresponding energy functional is finite.

The proof we will give for the uniqueness of the exterior gravitational field consists in two steps. First, we will show that two solutions with the same Dirichlet boundary conditions on $\tilde{\Sigma}$ must coincide. To that aim, let us assume that there are two harmonic maps Ψ_0 and Ψ_1 from M to N satisfying $U_0|_{\tilde{\Sigma}} = U_1|_{\tilde{\Sigma}}, \quad \Omega_0|_{\tilde{\Sigma}} = \Omega_1|_{\tilde{\Sigma}}$. To proof that they coincide everywhere, we will make use of a most remarkable identity due to Bunting [11]. The idea is to construct a smooth homotopic application

$$\begin{aligned} \Psi : \quad & M \times [0,1] \longrightarrow N \\ & (x, u) \longrightarrow \Psi(x, u) \end{aligned} \tag{16}$$

satisfying $\Psi(x, 0) = \Psi_0$, $\Psi(x, 1) = \Psi_1$ everywhere on M and such that $\gamma_{x_0}(u) \equiv \Psi(x_0, u)$ is an affinely parametrized geodesic in N for every $x_0 \in M$. In our case, N is a simply connected geodesically complete maximally symmetric plane with constant

negative curvature so that a well-known theorem (see e.g. [12]) asserts that any two points can be joined by a unique geodesic. Therefore the homotopy (16) can always be constructed. A remarkably simple calculation (see also [13]) leads to Bunting's identity

$$\frac{1}{2}\nabla^{\mu}\nabla_{\mu}s^{2}=\int_{0}^{1}du\left\{\nabla^{\mu}s_{i}\nabla_{\mu}s^{i}-R_{ijkl}s^{i}\nabla_{\mu}\Psi^{j}s^{k}\nabla^{\mu}\Psi^{l}\right\} \tag{17}$$

where R_{ijkl} is the Riemann tensor in N, $s^i(x)$ is the tangent vector along the geodesic γ_x and $s^2(x)$ is its norm (being geodesic and affinely parametrized s^2 does not depend on u). For N with non-positive curvature, the righthand side of this identity is non-positive definite and it only vanishes when s^i is covariantly constant. The lefthand side in (17) is a total divergence and its integral in M can be transformed into a surface integral on the boundary $\tilde{\Sigma}$ and the integral "at infinity". The coincidence of Ψ_0 and Ψ_1 on $\tilde{\Sigma}$ implies $s|_{\tilde{\Sigma}}=0$ and consequently the surface integral on $\tilde{\Sigma}$ vanishes. Regarding the integral "at infinity", the asymptotic behaviour of Ψ_0 and Ψ_1 is of the form (13)

$$\Psi_D=\left(-m_D R^{-1}+O\left(R^{-2}\right),-2J_D\, zR^{-3}+O\left(R^{-3}\right)\right),\qquad D=0,1$$

(we cannot assume a priori the same mass and angular momentum for both solutions). The geodesics of N can be explicitly integrated so that the evaluation of the geodesic distance between $\Psi_0(x)$ and $\Psi_1(x)$ at points x lying on a spherical surface $R=R_0$ proves that s falls off much faster than R_0^{-2} when R_0 tends to infinity. Hence, the integral on the surface "at infinity" also vanishes and we can conclude that s^i is a constant vector field in M. Since s^i vanishes on $\tilde{\Sigma}$, s^i must be zero everywhere and $\Psi_0(x)=\Psi_1(x)\ \ \forall x\in M$ as we wanted to prove.

In the second part of the proof we must take into account that, as shown above, the matching conditions fix the value of Ω on Σ (and thereby on $\tilde{\Sigma}$) up to an arbitrary additive constant *which is relevant*, because the freedom in Ω has already been used to set the asymptotic behaviour (13). Therefore, we still need to show that this arbitrary additive constant does not introduce a one-parameter family of solutions. To that end, the full set of boundary conditions (12) on U and Ω will now help. Let us assume there exist two solutions $\Psi_+=(U_+,\Omega_+)$, and $\Psi_-=(U_-,\Omega_-)$ satisfying

$$U_+|_{\tilde{\Sigma}}=U_-|_{\tilde{\Sigma}}\,,\ \ \nabla_\mu U_+|_{\tilde{\Sigma}}=\nabla_\mu U_-|_{\tilde{\Sigma}}\,,\ \ \Omega_+|_{\tilde{\Sigma}}=\Omega_-|_{\tilde{\Sigma}}+C,\ \ \nabla_\mu\Omega_+|_{\tilde{\Sigma}}=\nabla_\mu\Omega_-|_{\tilde{\Sigma}} \tag{18}$$

where C is an arbitrary constant. A direct calculation shows that, given any solution $\Psi=(U,\Omega)$ of the Ernst equations (4), the three-parameter one-form on M

$$\boldsymbol{W}\left(\Psi;c_1,c_2,c_3\right)\equiv 2\left(c_2+2c_3\Omega\right)dU+\left[e^{-4U}\left(c_1+c_2\Omega+c_3\Omega^2\right)-c_3\right]d\Omega$$

has vanishing divergence, (i.e. $\nabla^\mu W_\mu\left(\Psi\right)\equiv 0$) and, consequently,

$$\int_{\tilde{\Sigma}}W_\mu dS^\mu=\int_{S_\infty}W_\mu dS^\mu=8\pi c_2 m(\Psi) \tag{19}$$

where S_∞ stands for the surface "at infinity" and the last equality follows immediately from the asymptotic flatness conditions (13). On the other hand, relations (18) imply

$$\boldsymbol{W}\left(\Psi_+;c_1,c_2,c_3\right)|_{\tilde{\Sigma}} = \boldsymbol{W}\left(\Psi_-;c_1+c_2C+c_3C^2,c_2+2c_3C,c_3\right)\Big|_{\tilde{\Sigma}}.$$

The combination of this expression with the identity (19) produces

$$8\pi c_2 m(\Psi_+) = 8\pi\left(c_2+2c_3C\right)m(\Psi_-)$$

and, given that c_2 and c_3 are arbitrary constants, it necessarily follows

$$m\left(\Psi_+\right) = m\left(\Psi_-\right), \qquad m(\Psi_-)C = 0.$$

Since for physically well-behaved solutions the total mass cannot vanish, C must be zero and the uniqueness of the exterior solution generated by a given interior distribution of matter in stationary and axially symmetric rotation is completely proven.

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